Spectrum for $Y=0$ brane in planar AdS/CFT

Ryo Suzuki (ITF, Utrecht University)
with Zoltán Bajnok (Hungarian Academy of Science)
Raphael Nepomechie (Univ. Miami)
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Based on JHEP 1208 (2012) 149
October 2012
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AdS/CFT for open and closed strings
AdS/CFT Correspondence

IIB string on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills should make the same prediction in the large $N$ limit

with the identification

$$\frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi \alpha'} \sim \sqrt{N} g_{\text{str}} \iff \lambda = Ng_{\text{YM}}^2$$
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Strong Weak Duality

Semiclassical string $\lambda \gg 1$ SYM perturbation $\lambda \ll 1$

- Difficulty if we want to study AdS/CFT
- Advantage if we want to apply AdS/CFT
AdS/CFT Correspondence

IIB string on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4 \ SU(N)$ super Yang-Mills should make the same prediction in the large $N$ limit

with the identification $\frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi \alpha'} \sim \sqrt{Ng_{\text{str}}} \leftrightarrow \lambda = Ng_{\text{YM}}^2$

Strong Weak Duality

Semiclassical string $\lambda \gg 1$ SYM perturbation $\lambda \ll 1$

Integrability + superconformal symmetry

• Possible to test AdS/CFT by the exact computation

Tuesday, November 13, 2012
Most studied physical observables in AdS/CFT are

**Closed string states** ↔ **Single-trace operators**

Energy of a short spinning string

$\mathcal{E}(\lambda)$

Dimension of Konishi multiplet

\[
\text{tr}(\Phi^I \Phi^I) \\
\text{tr}(Z^2 W^2 - (ZW)^2) \\
\text{tr}\left(D_+ Z^2 - (D_+ Z)^2\right)
\]

$\Delta(\lambda)$

$W \equiv \Phi^1 + i\Phi^2, \quad Y \equiv \Phi^3 + i\Phi^4, \quad Z \equiv \Phi^5 + i\Phi^6$
Most studied physical observables in AdS/CFT are:

**Closed string states** ↔ **Single-trace operators**

- Energy of a short spinning string
- Dimension of Konishi multiplet
  \[ \text{tr} \left( \Phi^I \Phi^I \right) \]
  \[ \text{tr} \left( Z^2 W^2 - (ZW)^2 \right) \]
  \[ \text{tr} \left( D_+ Z^2 - (D_+ Z)^2 \right) \]

**Energy of a periodic spin chain state**

\[ E(\lambda) \]

\[ \Delta(\lambda) \]

Exact spectrum via TBA

Tuesday, November 13, 2012
The exact Konishi dimension

- **SYM results up to 5-loop**
  [Fiamberti, Santambrogio, Sieg, Zanon (2007)] [Velizhanin (2008)]
  [Eden, Heslop, Korchemsky, Smirnov, Sokatchev (2012)]

- **String results up to 1-loop**

- **Numerical results up to \( l \approx 2000 \)**
  [Gromov, Kazakov, Vleira (2009)] [Frolov (2010)] and others

- **Analytic results up to 7-loop at weak coupling**
Open string sector in AdS/CFT are less studied

- Minimal surface vs. Wilson loop vev
  An open string (or disk worldsheet) ending on a stack of N D3 branes

- Spectrum of open string state vs. Determinant-like operators
  An open string ending on another rotating single D(3)-brane
Open string sector in AdS/CFT are less studied

- Minimal surface vs. Wilson loop vev

An open string (or disk worldsheet) ending on a stack of $N$ D3 branes

**This Talk**

- Spectrum of open string state vs. Determinant-like operators

An open string ending on another rotating single D(3)-brane

$= \text{(Spherical) Giant gravitons}$
• **Determinant operators correspond to D-branes (without open string)**

\[
\det Z \equiv \varepsilon_{i_1 \ldots i_N} \varepsilon^{j_1 \ldots j_N} Z_{j_1}^{i_1} \ldots Z_{j_N}^{i_N}
\]

\[
\text{Half BPS} \quad \Rightarrow \quad \Delta_{\text{det}} = N
\]

• **Determinant-like operators correspond to D-branes with open string excitations**

\[
\mathcal{O}_1 \equiv \varepsilon_{i_1 \ldots i_N} \varepsilon^{j_1 \ldots j_N} Z_{j_1}^{i_1} \ldots Z_{j_{N-1}}^{i_{N-1}} \chi_{j_N}^{i_N}
\]

\[
\mathcal{O}_2 \equiv \varepsilon_{i_1 \ldots i_N} \varepsilon^{j_1 \ldots j_N} Y_{j_1}^{i_1} \ldots Y_{j_{N-1}}^{i_{N-1}} \chi_{j_N}^{i_N}
\]

\[
\text{Non-BPS} \quad \Rightarrow \quad \Delta[\mathcal{O}_{1,2}] - N \text{ is nontrivial}
\]
Giant graviton is determinant

- Matching of the residual symmetry
  \[
  \text{det } Z \leftrightarrow S^3 \subset S^5 : SO(6) \to SO(4) \times SO(2)
  \]

- However, multi-traces may also be good because
  ✓ For large operators, multi-traces can mix at large \(N\)
  ✓ Determinant is a linear combination of multi-traces
  \[
  \text{det } Z = c[1^N](\text{tr } Z)^N + \cdots + c[N]\text{tr } Z^N, \quad c[x] = \text{constant}
  \]

- Determinant and sub-determinant do not correlate, nor do maximal and non-maximal giant gravitons

[Witten (1998)] [Balasubramanian, Berkooz, Naqvi, Strassler (2001)] [Corley, Jevicki, Ramgoolam (2001)]
Open string in AdS/CFT from integrability

Energy of open string ending on the D3-brane

\[ E(\lambda) \]

(Subtracted) dimension of determinant-like operator

\[ \Delta(\lambda) \]

One-loop Hamiltonian is integrable
Open string in AdS/CFT from integrability

Energy of open string ending on the D3–brane $E(\lambda)$

(Subtracted) dimension of determinant–like operator $\Delta(\lambda)$

Integrability Method

Energy of an open spin chain state with integrable boundary conditions

Exact spectrum via boundary TBA?
Why boundary?

- New examples of AdS/CFT dictionary by applying integrability methods (TBA/Y-system ...)
- Challenge to study more general integrable models (periodic → twist → deformation → boundary ...)
- Boundary models are intrinsically finite-size (c.f. Casimir effects between parallel plates)
Our goal and strategy

Want to compute the spectrum of an open string ending on the “Y=0” brane

- Boundary Bethe-Yang equations
  (Asymptotic Bethe Ansatz equations)

- Finite-size corrections (Lüscher formula)

- Conjecture the exact method (TBA/Y-system)

[Correa, Young (2009)] [Bajnok, Palla (2010)]

[Hofman, Maldacena (2007)]

[Galleas (2009)]

[Bajnok, Nepomechie, Palla, RS (2012)]
Our goal **and strategy**

Want to compute the spectrum of an open string ending on the “$Y=0$” brane

- **Boundary Bethe-Yang equations**
  - *(Asymptotic Bethe Ansatz equations)*

- **Finite-size corrections** *(Lüsher formula)*

- Conjecture the **exact** method *(TBA/Y-system)*

By conjecturing how to include integrable boundaries from the lessons in periodic (closed string) cases

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[Correa, Young (2009)] [Bajnok, Palla (2010)]

[Hofman, Maldacena (2007)]

[Correa, Young (2009)] [Bajnok, Palla (2010)]

[Galleas (2009)]

[Bajnok, Nepomechie, Palla, RS (2012)]
Plan of Talk

• AdS/CFT for open and closed strings
• Double-row transfer matrix
• The Y=0 brane
• Finite-size corrections from Lüscher formula
• Boundary Y-system and boundary TBA
• Conclusion
Integrable models with boundary:
double-row transfer matrix
Integrability in the $\sigma$-model on AdS$_5 \times S^5$

- This model is **classically integrable** because the target space is a supercoset.
- We break conformal symmetry by a gauge choice.
- By taking the **large-radius limit**, we can define asymptotic states and their S-matrix.
- This worldsheet S-matrix is (hopefully) **integrable**.
What is integrability?

Integrable S-matrices satisfy the Yang-Baxter relation

\[ S_{123} = S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12} \]

\[ S_{ij} : V_i \otimes V_j \rightarrow V_j \otimes V_i , \quad \text{act trivially on } V_k \ (k \neq i, j) \]
Integrable S-matrices satisfy the Yang-Baxter relation

What is integrability?

Integrable S-matrices factorize into the product of two-body S-matrices with any ordering of the product.

\[ S_{123} = S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12} \]

Many-body S-matrix factorizes into the product of two-body S-matrices with any ordering of the product.
Integrability and Yang–Baxter relation

Yang–Baxter tells that transfer matrices commute

\[ T_a(q) = (s) \text{tr}_{V_a} \left[ S_{a1}(q, p_1) \cdots S_{aN}(q, p_N) \right] \]

\[ T_a = S_{a1} \cdots S_{aN} : V_a \otimes V^\otimes N \rightarrow V^\otimes N \otimes V_a \]

\[ T_a : V^\otimes N \rightarrow V^\otimes N, \text{ matrix of dim } V^N \]
Integrability and Yang–Baxter relation

Yang–Baxter tells that transfer matrices commute

Yang-Baxter algebra: $S_{ab} T_a T_b = T_b T_a S_{ab}$

Take trace in $V_a \otimes V_b \Rightarrow [T(q_a), T(q_b)] = 0$

$T_a(q) = \sum_n Q_n q^n$ generates conserved charges $\{Q_n\}$
Summary of integrability

- Yang-Baxter relation (or algebra)
- Factorized S-matrix
- Transfer matrix generates infinite charges

Transfer matrix is an important quantity in (periodic) integrable models
Summary of boundary integrability

- **Boundary** Yang-Baxter relation (or algebra)
- **Integrable** reflection amplitude
- **Double-row** transfer matrix generates infinite charges

Double-row transfer matrix is important in boundary integrable models
Boundary Yang–Baxter relation

To maintain the integrability at boundary, boundary reflection and bulk scattering must commute [Sklyanin (1988)]

\[
S(-p_2, -p_1) \mathcal{R}(p_1) S(p_1, -p_2) \mathcal{R}(p_2) = \mathcal{R}(p_2) S(p_2, -p_1) \mathcal{R}(p_1) S(p_1, p_2)
\]

By using \( S(a, b) = S(-b, -a) \) this becomes

\[
S(p_1, p_2) \mathcal{R}(p_1) S(p_1, -p_2) \mathcal{R}(p_2) = \mathcal{R}(p_2) S(p_1, -p_2) \mathcal{R}(p_1) S(p_1, p_2)
\]
Boundary Yang–Baxter relation leads to

Boundary Yang–Baxter algebra

\[ S(p_1, p_2) T(p_1) S(p_1, -p_2) T(p_2) = T(p_2) S(p_1, -p_2) T(p_1) S(p_1, p_2) \]

However, we cannot just take the trace!

\[ T(p_1) T(p_2) \neq T(p_2) T(p_1) \]
Sklyanin combined the right- and left-reflections

\[ S_{12} T_1^- \tilde{S}_{12} T_2^- = T_2^- \tilde{S}_{12} T_1^+ S_{12} \]

\[ S_{12}^{-1} T_1^{+t_1} \tilde{S}_{12}^{-1} T_2^{+t_2} = T_2^{+t_2} \tilde{S}_{12}^{-1} T_1^{+t_1} S_{12}^{-1} \]

If the S-matrix is transpose invariant \( S_{12}^{t_1} = S_{12}^{t_2} \)

\[ D(q) \equiv \text{tr} \left[ T_-(q) T_+(q) \right] \quad \text{with different } q \text{ commute!} \]

Thus D generates infinite conserved charges
Double-row transfer matrix

\[ D_a = \text{tr}_a \left[ T_- T_+ \right] = \text{tr}_a \left[ S_{aN} \cdots S_{a1} R^- S_{1a} \cdots S_{Ny} R^+ \right] \]

- \( D_a \) is not the “square” of transfer matrix

\[ S_{aj} : V_a \otimes V_j \to V_j \otimes V_a, \quad S_{ja} : V_j \otimes V_a \to V_a \otimes V_j \]

\( S_{aj} S_{ja} \) is a matrix product
Double-row transfer matrix

\[ D_a = \text{tr}_a \left[ T_- T_+ \right] = \text{tr}_a \left[ S_{aN} \cdots S_{a1} R^{-} S_{1a} \cdots S_{Na} R^{+} \right] \]

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\( S_{aj} S_{ja} \) is a matrix product
Summary of boundary integrability

- Boundary Yang-Baxter relation (or algebra)
- Integrable reflection amplitude
- Double-row transfer matrix generates infinite charges

**Double-row transfer matrix** is important in boundary integrable models
The Y=0 brane
Spherical maximal giant gravitons (SMGG)

D3-brane in $\text{AdS}_5 \times S^5$
with a large angular momentum $J = \mathcal{O}(N)$

Spherical $\Leftrightarrow$ “wrap” on $S^3 \subset S^5$
with the angular momentum bound $J \leq N$

Maximal $\Leftrightarrow$ $J = N$ $\Leftrightarrow$ half-BPS state

Spherical maximal giant gravitons are dual to determinants

$\det \Phi \sim \epsilon^{i_1 \ldots i_N} \epsilon_{j_1 \ldots j_N} \Phi^{j_1}_{i_1} \ldots \Phi^{j_N}_{i_N}$

Open strings on SMGG are dual to determinant-like operators

$\mathcal{O}_\Phi (\chi) \sim \epsilon^{i_1 \ldots i_N} \epsilon_{j_1 \ldots j_N} \Phi^{j_1}_{i_1} \ldots \chi^{j_m}_{i_m} \ldots \Phi^{j_N}_{i_N}$
Spherical maximal giant gravitons (SMGG)  

[McGreevy, Susskind, Toumbas (2000)]

D3-brane in $\text{AdS}_5 \times S^5$

with a large angular momentum $J = \mathcal{O}(N)$

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**Spherical maximal giant gravitons are dual to determinants**

[Balasubramanian, Berkooz, Naqvi, Strassler (2001)]

$$\det \Phi \sim \epsilon^{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} \Phi_{i_1}^{j_1} \cdots \Phi_{i_N}^{j_N}$$

**Open strings on SMGG are dual to determinant-like operators**

[Balasubramanian, Huang, Levi, Naqvi (2002)]

$$\mathcal{O}_\Phi (\chi) \sim \epsilon^{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} \Phi_{i_1}^{j_1} \cdots \chi_{i_m}^{j_m} \cdots \Phi_{i_N}^{j_N}$$

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Classification of giant graviton branes

SMGG are classified according to the choice:

\[ S^3 \subset S^5 = \{ |X|^2 + |Y|^2 + |Z|^2 = R^2 \} \]

\[ X = 0 \text{ or } Y = 0 \text{ or } Z = 0 \ldots \]

SMGG as a boundary condition for a spin chain

\[
\begin{align*}
\text{tr} \ (ZZ \cdots ZZ) & \quad \text{Periodic} \\
\epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} Y^{N-1} (ZZ \cdots ZZ)_{j_N}^{i_N} & \quad Y = 0 \\
\epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} Z^{N-1} (ZZ \cdots ZZ)_{j_N}^{i_N} & \quad Z = 0
\end{align*}
\]

Insert \( Z^J \) to det \( \Phi \). The choice \( Z^J \) breaks the global symmetry

\[ \text{psu}(2,2|4) \rightarrow \text{psu}(2|2)^2 \times u(1) \]

which may be broken further by boundary conditions.
Classification of giant graviton branes

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SMGG as a boundary condition for a spin chain

$$\text{tr} \ (ZZ \cdots ZZ)$$

$$\epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} Y^{N-1} (ZZ \cdots ZZ)^i_{j_N}$$

$$\epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} Z^{N-1} (ZZ \cdots ZZ)^i_{j_N}$$

Periodic

$$Y = 0$$

$$Z = 0$$

Insert $Z^J$ to det $\Phi$. The choice $Z^J$ breaks the global symmetry $\text{psu}(2, 2|4) \rightarrow \text{psu}(2|2)^2 \times u(1)$

which may be broken further by boundary conditions

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Classification of giant graviton branes

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SMGG as a boundary condition for a spin chain

\[ \text{tr} (ZZ \cdots ZZ) \]
\[ \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} Y^{N-1} (ZZ \cdots ZZ)^{i_N}_{j_N} \quad \text{Periodic} \]
\[ \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} Z^{N-1} (ZZ \cdots ZZ)^{i_N}_{j_N} \quad Y = 0 \]
\[ \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} \quad Z = 0 \]

Insert \( Z^J \) to \( \det \Phi \). The choice \( Z^J \) breaks the global symmetry

\[ \text{psu}(2,2|4) \rightarrow \text{psu}(2|2)^2 \times u(1) \]

which may be broken further by boundary conditions
The $Y=0$ and $Z=0$ branes

Open string state on the $Y=0$ brane should correspond to

$$\mathcal{O}_Y(\chi) \sim \sum_k \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} Y^i_{j_1} \ldots Y^i_{j_{N-1}} (Z^k \chi Z^{j-k})^i_{j_N}$$

Open string state on the $Z=0$ brane should correspond to

$$\mathcal{O}_Z(\chi, \chi', \chi'') \sim \sum_k \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} Z^i_{j_1} \ldots Z^i_{j_{N-1}} (\chi Z^k \chi' Z^{j-k} \chi'')^i_{j_N}$$

Unlike spinning strings, giant gravitons extends along the axis of rotation; like a electric dipole moving in the magnetic flux
The $Y=0$ and $Z=0$ branes

Open string state on the $Y=0$ brane should correspond to

$$\mathcal{O}_Y(\chi) \sim \sum_k \epsilon_{i_1 \ldots i_N} \epsilon_{j_1 \ldots j_N} Y_{j_1}^{i_1} \ldots Y_{j_{N-1}}^{i_{N-1}} (Z^k \chi Z^{J-k}) j_N^{i_N}$$

Open string state on the $Z=0$ brane should correspond to

$$\mathcal{O}_Z(\chi, \chi', \chi'') \sim \sum_k \epsilon_{i_1 \ldots i_N} \epsilon_{j_1 \ldots j_N} Z_{j_1}^{i_1} \ldots Z_{j_{N-1}}^{i_{N-1}} (\chi Z^k \chi' Z^{J-k} \chi'') j_N^{i_N}$$

Unlike spinning strings, giant gravitons extends along the axis of rotation; like a electric dipole moving in the magnetic flux

[McGreevy, Susskind, Toumbas (2000)]
The $Y=0$ branes

\[ \mathcal{O}_Y(\chi) \sim \sum_k \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} Y^{i_1}_{j_1} \ldots Y^{i_{N-1}}_{j_{N-1}} (Z^k \chi Z^J)^{i_N}_{j_N} \]

Preserves the symmetry $\mathfrak{psu}(1|2)^2$

No boundary degrees of freedom

\[ [\mathcal{R}_Y, J] = 0, \quad \forall J \in \mathfrak{psu}(1|2) \quad \Rightarrow \quad \mathcal{R}_Y \text{ is diagonal} \]

\[ \mathcal{R}_Y^{-}(p) = R_0^{-}(p)^2 \begin{pmatrix} e^{-ip/2} & \quad -e^{ip/2} \\ e^{ip/2} & \quad 1 \end{pmatrix} \otimes 2 \]

Tuesday, November 13, 2012
The $Z=0$ branes

\[ \mathcal{O}_Z(\chi, \chi', \chi'') \sim \sum_k \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} Z_{j_1}^{i_1} \ldots Z_{j_{N-1}}^{i_{N-1}} (\chi Z^k \chi' Z^{J-k} \chi'')_{j_N}^{i_N} \]

Preserves the symmetry $\mathfrak{psu}(2|2)^2$

Boundary degrees of freedom $\chi, \chi''$

(The determinant factorizes if $\chi, \chi'' = Z$)

$\mathbb{R}^-_Z : V(p) \otimes V_B \rightarrow V(-p) \otimes V_B \quad (p > 0)$

$\mathbb{R}^+_Z : V(p) \otimes V_B \rightarrow V(-p) \otimes V_B \quad (p < 0)$

The reflection amplitude $\mathbb{R}_Z$ is non-diagonal

Its matrix structure can be determined by the symmetry

[Hofman, Maldacena (2007)]
Boundary dressing phase

Reflection amplitude for the $Y=0$ brane

$$\mathbb{R}_Y^-(p) = R_0^-(p)^2 \left( \begin{array}{cc} e^{-i\frac{p}{2}} & -e^{i\frac{p}{2}} \\ 1 & 1 \end{array} \right) \otimes 2$$

The scalar factor is fixed by requiring that the total scattering phase of the singlet state is trivial after crossing [Beisert (2005)] [Hofman, Maldacena (2007)]

$$\Rightarrow \text{Boundary crossing equation}$$

$$R_0^-(p)^2 R_0^-(p)^2 = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}} \sigma(p, -p)^2$$

A solution consistent with various limits

$$R_0^-(p)^2 = -e^{-ip} \sigma(p, -p)$$

[Chen, Correa (2007)]
Finite-Size corrections from Lüscher formula
Bethe Yang equations

- Transfer matrix is related to Bethe Yang equations, whose solution captures the asymptotic energy

\[-1 = e^{-i J q} T(q | \vec{p}) \bigg|_{q=p_k} \iff -1 = e^{-i J p_k} \prod_{j=1}^{N} S(p_k, p_j)\]

\[E_{\text{asymptotic}} = \sum_{i=1}^{N} \sqrt{Q_i^2 + 4g^2 \sin^2 \frac{p_i}{2}}, \quad g = \frac{\sqrt{\lambda}}{2\pi}\]
Boundary Bethe Yang equations

- Double-row transfer matrix is related to **Boundary Bethe Yang equations**, whose solution captures the asymptotic energy

\[
-1 = e^{-2iqJ} D(q|\vec{p}) \bigg|_{q=p_k} \Leftrightarrow
\]

\[
-1 = e^{-i2Jp_K} \prod_{j=1}^{N} S(p_k, p_j) R^{-}(p_k) \prod_{j=1}^{N} S(p_j, -p_k) R^{+}(-p_k)
\]

\[
E_{\text{asymptotic}} = \sum_{i=1}^{2N} \sqrt{Q_i^2 + 4g^2 \sin^2 \frac{p_i}{2}}
\]
• Bethe-Yang equations determine the asymptotic spectrum of closed string

• Boundary Bethe-Yang equations determine the asymptotic spectrum of open string

• Finite J corrections come from virtual particles in the mirror kinematics

\[
\sum_Q \int d\mathcal{E}_Q \int d\tilde{p}_Q \, e^{-i\mathcal{E}_Q J} \sim \sum_Q \int d\tilde{p}_Q \, e^{-\tilde{\mathcal{E}}_Q (\tilde{p}_Q)^J}
\]

\[(\mathcal{E}_Q, p_Q) = (-i\tilde{p}_Q, -i\tilde{\mathcal{E}}_Q), \quad \tilde{\mathcal{E}}_Q = 2 \text{arcsinh} \left( \frac{\sqrt{Q^2 + \tilde{p}_Q^2}}{2g} \right)\]
Finite-size corrections to closed spectrum

- Lüscher formula was the main tool to study the finite-size corrections to the closed string spectrum

\[ \delta E \simeq \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\tilde{p}_Q}{2\pi} Y_Q^o, \quad Y_Q^o = e^{-\tilde{\epsilon}_Q J} T_Q^2 \]

- Lüscher formula is written in terms of transfer matrices

\[ \delta E = - \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\tilde{p}_Q}{2\pi} Y_Q^o \]

Sum over virtual particles

[Lüscher (1986)] [Janik Łukowski (2007)]
Finite-size corrections to closed spectrum

- Lüscher formula was the main tool to study the finite-size corrections to the closed string spectrum

\[ \delta E \simeq \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\tilde{\rho}_Q}{2\pi} Y_Q^0, \quad Y_Q^0 = e^{-\tilde{\epsilon}_Q J} T_Q^2 \]

Written in terms of transfer matrices

Sum over virtual particles
Finite-size corrections to open spectrum

- Boundary Lüscher formula has been conjectured and tested

\[ \delta E \sim \]

\[ \delta E = - \sum_{Q=1}^{\infty} \int_{0}^{\infty} \frac{d\tilde{\rho}_Q}{2\pi} Y_Q^\circ, \quad Y_Q^\circ = e^{-2\tilde{\varepsilon}_Q J} D_Q^2 \]

- Written in terms of double-row transfer matrices

Sum over virtual particles
Finite-size corrections to open spectrum

- Boundary Lüscher formula has been conjectured and tested

\[ \delta E \simeq \sum_{Q=1}^{\infty} \int_{0}^{\infty} \frac{d\tilde{\rho}_Q}{2\pi} Y_Q^o, \quad Y_Q^o = e^{-2\tilde{E}_Q J} D_Q^2 \]

- Written in terms of double-row transfer matrices
Prediction of boundary Lüscher formula

• The Y=0 ground state is BPS. Since its energy is protected, finite-size corrections vanish.

\[ \delta E[\mathcal{O}_Y(1)] = 0 \]  
[Correa, Young (2009)]

• The finite-size corrections to the energy of Y=0 single-particle states are nontrivial.

\[ \delta E[\mathcal{O}_Y(Y)] \approx g^{12} \cdot 192 (4\zeta_5 - 7\zeta_9), \text{ for } (J, n) = (2, 1) \]  
[Bajnok, Palla (2010)]

This is six-loop results in N=4 SYM. Field theoretical computation has been performed for Z=0 at four loop, but not Y=0. [Correa, Young (2009)]

\[ \mathcal{O}_Y(\chi) \sim \sum_k \epsilon_{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} Y^{i_1}_{j_1} \ldots Y^{i_{N-1}}_{j_{N-1}} (Z^k \chi Z^{J-k})^{i_N}_{j_N} \]
Prediction of boundary Lüscher formula

• For general $Y=0$ multi-particle states, we need to diagonalize $D_Q$ by means of algebraic Bethe Ansatz

[Arutyunov, de Leeuw, RS, Torrielli (2009)] [Galleas (2009)]

• However, the computation of the fully general case is too complicated to perform

• We conjecture the generating function for the eigenvalues of $D_Q$ as in the periodic case

[Beisert (2006)] [Bajnok, Nepomechie, Palla, RS (2012)]
Generating function for the eigenvalues of $D_Q$

The su(2) sector, case of $Q=1$  [Galleas (2009)]

\[ D_1 = \rho_1 \Lambda_1 + \rho_2 \Lambda_2 - \rho_3 \Lambda_3 - \rho_4 \Lambda_4 \]

**Bulk factor**
\[ \Lambda_1 = 1, \quad \Lambda_2 = \frac{\mathcal{R}(-)+ \mathcal{B}(-)-}{\mathcal{R}(+)+ \mathcal{B}(+)-}, \quad \Lambda_3 = \Lambda_4 = \frac{\mathcal{R}(-)+}{\mathcal{R}(+)+} \]

**Boundary factor**
\[ \rho_1 = \rho_3 = \frac{(1 + (x^-)^2)(x^- + x^+)}{2x^+(1 + x^+x^-)}, \quad \rho_2 = \rho_4 = \frac{x^-(x^- + x^+)(1 + (x^+)^2)}{2(x^+)^2(1 + x^-x^+)} \]

**Notation:**
\[ \mathcal{R}^{(\pm)} = \prod_{i=1}^{N} (x(p) - x^{\mp}(p_i)) (x(p) - x^{\mp}(-p_i)), \quad \mathcal{B}^{(\pm)} = \prod_{i=1}^{N} \left( \frac{1}{x(p)} - x^{\mp}(p_i) \right) \left( \frac{1}{x(p)} - x^{\mp}(-p_i) \right) \]

\[ x(u) + \frac{1}{x(u)} = \frac{u}{g}, \quad p_Q(u) = -i \log \frac{x[+Q]}{x[-Q]}, \quad f[n](u) = f\left(u + \frac{in}{2}\right) \]

\[ g = \frac{\sqrt{\lambda}}{2\pi} \text{ is coupling constant, } x = x(u) \text{ or } x = x(p) \]
Generating function for the eigenvalues of $D_Q$

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\[ \Lambda_1 = 1, \quad \Lambda_2 = \frac{\mathcal{R}^{(-)+} + \mathcal{B}^{(-)-}}{\mathcal{R}^{(+)+} + \mathcal{B}^{(+)-}}, \quad \Lambda_3 = \Lambda_4 = \frac{\mathcal{R}^{(-)+}}{\mathcal{R}^{(+)+}} \]

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Generating function for the eigenvalues of $D_Q$

By using the eigenvalue of $Q=1$

$$D_1 = \rho_1 \Lambda_1 + \rho_2 \Lambda_2 - \rho_3 \Lambda_3 - \rho_4 \Lambda_4$$

the generating function for general $Q$ is given by

$$\tilde{\mathcal{W}}^{-1} = (1 - \mathcal{D}\rho_1 \Lambda_1 \mathcal{D})(1 - \mathcal{D}\rho_3 \Lambda_3 \mathcal{D})^{-1}(1 - \mathcal{D}\rho_4 \Lambda_4 \mathcal{D})^{-1}(1 - \mathcal{D}\rho_2 \Lambda_2 \mathcal{D}) = \sum_Q (-1)^Q \mathcal{D}^Q \mathcal{D}_Q \mathcal{D}^Q$$

where $\mathcal{D} = e^{-\frac{i}{2} \partial_u}$ $\iff$ $\mathcal{D} f(u) = f^-(u) \mathcal{D}$
Generating function for the eigenvalues of $D_Q$

By using the eigenvalue of $Q=1$

$$D_1 = \rho_1 \Lambda_1 + \rho_2 \Lambda_2 - \rho_3 \Lambda_3 - \rho_4 \Lambda_4$$

the generating function for general $Q$ is given by

$$\tilde{W}^{-1} = (1 - D\rho_1 \Lambda_1 D)(1 - D\rho_3 \Lambda_3 D)^{-1}(1 - D\rho_4 \Lambda_4 D)^{-1}(1 - D\rho_2 \Lambda_2 D)$$

$$= \sum_Q (-1)^Q D^Q D_Q D^Q$$

where $D = e^{-\frac{i}{2} \partial_u} \Leftrightarrow Df(u) = f^-(u)D$

$D_Q = D_{Q,1}$ corresponds to $Q$ symmetric rep. of $\mathfrak{psu}(2|2)$

$D_{1,Q}$ for $Q$ antisymmetric reps. of $\mathfrak{psu}(2|2)$ are generated by $\tilde{W}$

We checked $D_{1,1}$, $D_{2,1}$, $D_{1,2}$ by direct computation
• Using the generating function we predicted the finite-size corrections to the energy of various $Y=0$ (single-particle) states, e.g.

$$\delta E[\mathcal{O}_Y(X)] \approx -2^5 \cdot g^{20} \left[ -2^3 \cdot 7 \cdot (99 - 70\sqrt{2})\zeta_9 - 2(6765 - 4785\sqrt{2})\zeta_{11} - 2002(5\sqrt{2} - 7)\zeta_{15} + (7293 - 4862\sqrt{2})\zeta_{17} \right], \quad \text{for (}J, n\text{) = (2, 1)}$$

• The result can be generalized to the full sector of $\text{AdS}_5 \times \text{S}^5$

[Bajnok, Nepomechie, Palla, RS (2012)]
Boundary Y-system and boundary TBA
Generating function and T–system

\[ \tilde{W}^{-1} = \sum_a (-1)^a D^a D_{a,1} D^a, \quad \tilde{W} = \sum_s D^s D_{1,s} D^s \]

- The generated transfer matrices solve the su(2|2)\(^2\) T-system

\[ D_{a,s}^+ D_{a,s}^- = D_{a-1,s} D_{a+1,s} + D_{a,s-1} D_{a,s+1} \]

- We conjecture that they provide the asymptotic solutions of boundary TBA equations which gives the exact spectrum of Y=0 states

[Bajnok, Nepomechie, Palla, RS (2012)]
T–system and Y–system

The double–row transfer matrices satisfy asymptotic T–system

\[ T_{a,s}^+ T_{a,s}^- = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1} \]

Introduce Y–functions

\[ Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}} \]

Y–system

\[ \frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a-1,s} Y_{a+1,s}} = \frac{(1 + Y_{a,s+1})(1 + Y_{a,s-1})}{(1 + Y_{a-1,s})(1 + Y_{a+1,s})} \]

The same structure as in the closed string case!


Exact energy (for open strings)

\[ E_Q = \sum_{i=1}^{N} \left( \mathcal{E}_{Q_i}(p_i) + \mathcal{E}_{Q_i}(-p_i) \right) - \sum_{Q=1}^{\infty} \int_{0}^{\infty} \frac{d\tilde{p}_Q}{2\pi} \log(1 + Y_{Q,0}) \]
T-system and Y-system

The double-row transfer matrices satisfy asymptotic T-system

\[ T_{a,s}^+ T_{a,s}^- = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1} \]

Introduce Y-functions

\[ Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}} \]

Y-system

\[ \frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a-1,s} Y_{a+1,s}} = \frac{(1 + Y_{a,s+1})(1 + Y_{a,s-1})}{(1 + Y_{a-1,s})(1 + Y_{a+1,s})} \]
Mirror trick with boundary

- Mirror trick for periodic TBA

\[
Z_E(L, R) = \tilde{Z}_E(R, L) \to \exp(-L\mathcal{F}(R)), \quad R \to \infty
\]

Extremization condition for the “mirror” free energy is called TBA equations

Typically \( \log Y_a = V_a + \log(1 + Y_b) \ast K_{ba} \)
Mirror trick with boundary

- Mirror trick for boundary TBA

\[ \langle e^{-R \mathcal{H}_{\ell r}} \rangle = \langle B_\ell | e^{-L \tilde{\mathcal{H}}} | B_r \rangle = \sum_n \langle B_\ell | n \rangle e^{-L \tilde{\epsilon}_n} \langle n | B_r \rangle \]

Extremize the mirror free energy with the driving term

\[ V_{\ell, r} = \log \left( \langle B_\ell | n \rangle \langle n | B_r \rangle \right) \]

[Leclaire, Mussardo, Saleur, Skorik (1995)]

N.B. Such term often disappears when we derive Y-system from TBA
Mirror trick with boundary

- Problems to derive the boundary TBA

\[ V_{\ell,r} = \log (\langle B_{\ell}|n\rangle \langle n|B_{r}\rangle) \]

However, the boundary states \(|B_{\ell,r}\rangle\) are written in the Zamolodchikov-Faddeev basis instead of the Bethe Ansatz basis. These two bases are related non-trivially for the integrable models with non-diagonal S-matrix. Hence it is difficult to compute \(\langle n|B_{\ell,r}\rangle\) and to derive BTBA in the AdS/CFT setup.
From boundary Y-system to BTBA

- We may still conjecture BTBA for Y=0 brane
- BTBA should be same as the TBA for closed strings except for the source terms
- The source term can often be fixed by the asymptotic data
- In other words, we integrate (boundary) Y-system with (asymptotic) discontinuity relations to get/define BTBA
Exact energy for $Y = 0$ and $Y = 0 & \bar{Y} = 0$

- Since $Y=0$ brane is BPS, the exact ground state energy vanishes
- More interesting to study non-BPS ground states
  
  e.g. $Y = 0$ on the left, $\bar{Y} = 0$ on the right
  
- This corresponds to changing the supertrace to the trace
- Open tachyon in the spectrum

Konishi energy $E \approx 2\lambda^{1/4} = 2\frac{R}{\sqrt{\alpha'}}$

Open tachyon energy $E \approx -\lambda^{1/4}$

Need to solve BTBA numerically
Conclusion
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- Studied AdS/CFT for open strings ending on SMGG by using integrability methods
- Conjectured generating function for the double-row transfer matrix
- $Y$-system for $Y=0$ brane is same as $Y$-system for closed strings

Future directions

- Formulation of BTBA and numerical solution
- Small angle limit and analytic solution
- Rigorous derivation of integrability method
- $Z=0$ and other types of boundary conditions
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- Studied AdS/CFT for open strings ending on SMGG by using integrability methods
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Thank you for attention