SUSY Gauge Theory on Squashed Three-Spheres

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Naofumi Hama, Sungjay Lee and KH, arXiv:1102.4716,
Naofumi Hama, Sungjay Lee and KH, JHEP 1103:127,2011, arXiv: 1012.3512,
Overview

We computed partition functions ($Z$) of 3D $\text{N}=2$ SUSY gauge theory on squashed $S^3$, as functions of

1. coupling constants

2. axis-length parameters ($\ell, \tilde{\ell}$) of squashed $S^3$

As metric on squashed $S^3$ we take the familiar one,

$$\text{d} s^2 = \ell^2 (\mu^1 \mu^1 + \mu^2 \mu^2) + \tilde{\ell}^2 \mu^3 \mu^3$$

($\mu^a$ : Left-Invariant 1-forms)

as well as that of “hyper-ellipsoid”,

$$\text{d} s^2 = \ell^2 (dx_0^2 + dx_1^2) + \tilde{\ell}^2 (dx_2^2 + dx_3^2)$$

$$ (x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1)$$
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\((x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1)\)

\(Z\) depends on only \(\tilde{\ell}\).

\(Z\) depends on both \(\ell\) and \(\tilde{\ell}\).
General 3D N=2 SUSY gauge theory consists of

<table>
<thead>
<tr>
<th>* gauge multiplet : $(A_m, \lambda_\alpha, \sigma, D)$</th>
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<td>Spin : 1, 1/2, 0, 0 (aux.)</td>
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<th>* matter multiplet : $(\phi, \psi_\alpha, F)$ + c.c.</th>
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Strategy : Localization principle

* Path integral localizes onto “saddle points”
  (= SUSY invariant bosonic field configurations )
* For integration along the directions transverse to saddle point locus,
  Gaussian approx. is exact.

* Saddle points = Coulomb branch vacua
  
  $(\sigma = \text{const.}, \ D \sim \sigma, \text{all other fields vanish})$

No Higgs branch due to conformal mass.
Partition function reduces to an integral over Coulomb branch $\mathcal{M}_C$. Schematically

$$Z = \int_{\mathcal{M}_C} [d\sigma] \left( \frac{\det \Delta_F}{\det \Delta_B} \right) \exp \left( - S_{\text{cl}} \right)$$

1. Integral over Lie algebra of gauge symmetry. (can be further reduced to Cartan subalgebra)

2. “one-loop determinant” which results from Gaussian integration. (matter mass parameter enters here)

3. Classical value of Euclidean action (FI coupling, CS coupling enter here)
Contribution of **gauge** and **matter** multiplets to partition function.

* **gauge multiplet**: (gauge sym: $G'$)

\[
\frac{1}{|W|} \int_T d^r \sigma \prod_{\alpha \in \Delta_+} \sinh(\pi b \alpha \cdot \sigma) \sinh(\pi b^{-1} \alpha \cdot \sigma)
\]

$W$: Weyl group  \quad r$: rank

$T$: Cartan subalgebra  \quad \Delta_+$: positive root set

* **matter multiplet**: (sitting in the rep. $R$ of $G$, with R-charge $q$)

\[
\prod_{\rho \in R} s_b(i - iq - \rho \cdot \sigma - M)
\]

$\rho$: weight vector

\[
s_b(x) \equiv \prod_{m,n \geq 0} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix}, \quad Q = b + b^{-1}
\]
Note:

The expressions for the measure and the determinant appear as building blocks in structure constants of Liouville or Toda CFTs with coupling $b$.

(cf. Liouville central charge: $c_L = 1 + 6(b + b^{-1})^2$)

“3d version of AGT correspondence”

* For round $S^3$, $b = 1$.
* For squashed $S^3$ with (familiar) left-invariant metric, $b = 1$.
* For squashed $S^3$ with hyper-ellipsoidal metric, we found

$$b = \sqrt{\ell / \tilde{\ell}}$$
3d version of AGT: an example

Recall: Partition function on $S^4$ of N=2 SUSY gauge theory

\[ Z_{SW} = \int d\nu(\alpha) F_{m,\alpha(\tau)} F_{m,\alpha(\tau)} = \left\langle \prod_i V_{m_i} \right\rangle \]

SW theory $\leftrightarrow$ Punctured Riemann surface

gauge couplings $\leftrightarrow \tau \leftrightarrow$ cpx str.

matter masses $\leftrightarrow m \leftrightarrow$ external Liouville momenta

Coulomb moduli $\leftrightarrow \alpha \leftrightarrow$ Internal Liouville momenta

Let us take the example:

SU(2) N=2* SYM theory $\leftrightarrow$ Liouville torus 1pt function
In the presence of a Janus domain wall along the equator $S^3$ across which the gauge coupling jumps from $\tau$ to $\tau'$,

$$Z_{\text{Janus}} = \int d\nu(\alpha) \overline{F_{m,\alpha}(\tau)} F_{m,\alpha}(\tau')$$

Set $\tau' = -1/\tau$ and apply the S-duality to the S-hemisphere.

$$Z_{\text{Janus}} = \int d\nu(\alpha) \overline{F_{m,\alpha}(\tau)} F_{m,\alpha}(-1/\tau)$$

$$= \int d\nu(\alpha) d\nu(\beta) \overline{F_{m,\alpha}(\tau)} S_{\alpha,\beta,m} F_{m,\beta}(\tau)$$

$S_{\alpha,\beta,m}$ should correspond to the DOF on the “S-duality wall”.

For $N=2^*$ SU(2) SYM, the 3d theory on the wall is a certain $N=2$ SQED with mass & FI parameters $(\alpha, \beta, m)$.

[Gaiotto-Witten, Lee-Park-KH]
The partition function of the 3d wall theory:

\[ S_{\alpha, \beta, m} = \int_{\mathbb{R}} d\sigma e^{4\pi i\beta\sigma} s_b(-m) \]

\[ \times s_b\left(\frac{i}{2} + \frac{m}{2} + \alpha + \sigma\right)s_b\left(\frac{i}{2} + \frac{m}{2} + \alpha - \sigma\right) \]

\[ \times s_b\left(\frac{i}{2} + \frac{m}{2} - \alpha + \sigma\right)s_b\left(\frac{i}{2} + \frac{m}{2} - \alpha - \sigma\right) \]

... agrees precisely with the S-duality transformation coefficient of torus one-point Virasoro conformal blocks. [Teschner 2003]
Exact partition function for 3d gauge theories on sphere is another powerful tool for

* checking various 3d dualities

* determining the R-symmetry of IR superconformal fixed point theory ("Z-minimization")

* understanding the $O(N^{3/2})$ behavior of the DOF in multiple M2-brane theory
SUSY on $S^3$, round and squashed

In order for a curved space to support SUSY, it has to have **Killing spinors**

$$\mathcal{E} \text{ is a Killing spinor } \quad \nabla_\mu \mathcal{E} = \gamma_\mu \tilde{\mathcal{E}} \quad \text{for some } \tilde{\mathcal{E}}$$

$$\gamma_\mu = e_\mu^a \gamma^a, \quad \gamma^a : \text{Pauli matrix}$$

On round sphere,

**metric:** \( ds^2 = \ell^2 (\mu^1 \mu^1 + \mu^2 \mu^2 + \mu^3 \mu^3) \)

**vielbein:** \( e^a = \ell \mu^a, \quad \text{LI 1-forms: } g^{-1} dg = i \mu^a \gamma^a \)

there are 4 Killing spinors.

2 of them are constant in the “LI frame”, and satisfy

\[
\nabla_\mu \mathcal{E} = \frac{i}{2\ell} \gamma_\mu \mathcal{E}
\]

The other 2 are constant in the “RI frame”, and satisfy

\[
\nabla_\mu \mathcal{E} = -\frac{i}{2\ell} \gamma_\mu \mathcal{E}
\]
There are no Killing spinors on squashed $S^3$'s.

We turn on a suitable $U(1)$ background gauge field $V_\mu$, so that there are \textbf{charged} Killing spinors, satisfying

\[
\nabla_\mu \varepsilon = (\partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma^{ab} \mp iV_\mu) \varepsilon = \gamma_\mu \tilde{\varepsilon} \quad \text{for some } \tilde{\varepsilon}
\]

$\varepsilon$ carries $U(1)$ charge $\pm 1$

This $U(1)$ is the \textbf{R-symmetry} of 3d $N=2$ supersymmetry.
EX1. Squashed $S^3$, with SU(2)$_\text{Left}$-invariant metric

\[ ds^2 = \ell^2 (\mu^1 \mu^1 + \mu^2 \mu^2) + \tilde{\ell}^2 \mu^3 \mu^3, \]

\[ (e^1, e^2, e^3) = (\ell \mu^1, \ell \mu^2, \tilde{\ell} \mu^3) \]

One can show that any constant spinor $\varepsilon$ satisfies

\[ \frac{i}{2f} \gamma_{\mu} \varepsilon = \partial_\mu \varepsilon + \frac{1}{4} \gamma^{ab} \omega^{ab}_\mu \varepsilon - i V_\mu \gamma^3 \varepsilon, \]

\[ f \equiv \frac{\ell^2}{\tilde{\ell}}, \quad V_\mu \equiv \left( \frac{1}{\tilde{\ell}} - \frac{1}{f} \right) e^3_\mu \]

So, if the background U(1) gauge field $V_\mu$ is turned on,

\[ \varepsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] is a Killing spinor with U(1) charge +1

\[ \varepsilon = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \] is a Killing spinor with U(1) charge −1
EX2. Squashed $S^3$ with hyper-ellipsoidal metric

\[ ds^2 = f^2 d\theta^2 + \ell^2 \cos^2 \theta d\varphi^2 + \tilde{\ell}^2 \sin^2 \theta d\chi^2 \]

\[ f \equiv \sqrt{\ell^2 \sin^2 \theta + \tilde{\ell}^2 \cos^2 \theta} \]

\[ (e^1, e^2, e^3) = (\ell \cos \theta d\varphi, \tilde{\ell} \sin \theta d\chi, f d\theta) \]

\[ \psi_{\pm} \equiv \left( e^{\frac{i}{2}} (\pm \chi \mp \varphi + \theta) \right) \quad \text{satisfy} \]

\[ \frac{i}{2f} \gamma_\mu \psi_{\pm} = \left( \partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma^{ab} \mp i V_\mu \right) \psi_{\pm} \]

\[ V_\mu = \frac{1}{2} \left( \frac{\ell}{f} - 1 \right) d\varphi - \frac{1}{2} \left( \frac{\tilde{\ell}}{f} - 1 \right) d\chi \]
SUSY Theories on squashed $S^3$

For simplicity, let's consider free WZ model. We try

\[
\delta \phi = \bar{\epsilon} \psi, \\
\delta \psi = i \gamma^\mu \epsilon \nabla_\mu \phi + \bar{\epsilon} F, \\
\delta F = i \epsilon \gamma^\mu \nabla_\mu \psi,
\]

\[
\mathcal{L} = \nabla_\mu \bar{\phi} \nabla^\mu \phi - i \bar{\psi} \gamma^\mu \nabla_\mu \psi + \bar{F} F.
\]

We assume $\epsilon, \bar{\epsilon}$ are Killing spinors,

\[
\nabla_\mu \epsilon \equiv (\partial_\mu + \frac{1}{4} \omega^a_{\mu} \gamma^{ab} - i V_\mu) \epsilon = \frac{i}{2f} \gamma_\mu \epsilon,
\]

\[
\nabla_\mu \bar{\epsilon} \equiv (\partial_\mu + \frac{1}{4} \omega^a_{\mu} \gamma^{ab} + i V_\mu) \bar{\epsilon} = \frac{i}{2f} \gamma_\mu \bar{\epsilon},
\]

and check if $\delta \mathcal{L} = 0$. 

\[
\partial_\mu \text{ of flat space theory was replaced by } \nabla_\mu
\]
SUSY Theories on squashed $S^3$

For simplicity, let's consider free WZ model. We try

$$
\delta \phi = \bar{\epsilon} \psi,
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$$

$$
\delta F = i \epsilon \gamma^\mu \nabla_\mu \psi,
$$

$$
\mathcal{L} = \nabla_\mu \bar{\phi} \nabla^\mu \phi - i \bar{\psi} \gamma^\mu \nabla_\mu \psi + F F.
$$

$$
\delta \mathcal{L} = - \epsilon \bar{\psi} \cdot \nabla^\mu \nabla_\mu \phi - i ( - i \nabla_\mu \bar{\phi} \bar{\epsilon} \gamma^\mu + \bar{F} \epsilon ) \gamma^\nu \nabla_\nu \psi + i \bar{\epsilon} \gamma^\mu \nabla_\mu \psi \cdot F
$$

$$
\frac{\delta \mathcal{L}}{\delta \phi} = - \nabla^\mu \nabla_\mu \phi \cdot \bar{\epsilon} \psi + i \nabla_\mu \bar{\psi} \gamma^\mu ( i \gamma^\nu \epsilon \nabla_\nu \phi + \bar{\epsilon} F ) + \bar{F} \cdot i \epsilon \gamma^\mu \nabla_\mu \psi.
$$
SUSY Theories on squashed $S^3$

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$$

$$
\delta \mathcal{L} = - \epsilon \bar{\psi} \cdot \nabla^\mu \nabla_\mu \phi - \nabla_\mu \bar{\phi} \bar{\epsilon} \gamma^\mu \gamma^\nu \nabla_\nu \psi
$$

$$
- \nabla^\mu \nabla_\mu \bar{\phi} \cdot \bar{\epsilon} \psi - \nabla_\mu \bar{\psi} \gamma^\mu \gamma^\nu \epsilon \nabla_\nu \phi.
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$\partial_\mu$ of flat space theory was replaced by $\nabla_\mu$. 
For simplicity, let's consider free WZ model. We try

\[ \delta \phi = \bar{\epsilon} \psi, \]
\[ \delta \psi = i \gamma^\mu \epsilon \nabla_\mu \phi + \bar{\epsilon} F, \]
\[ \delta F = i \epsilon \gamma^\mu \nabla_\mu \psi, \]
\[ \mathcal{L} = \nabla_\mu \bar{\phi} \nabla^\mu \phi - i \bar{\psi} \gamma^\mu \nabla_\mu \psi + F \bar{F}. \]

Of flat space theory was replaced by \( \nabla_\mu \)

\[
\begin{align*}
\delta \mathcal{L} &= - \epsilon \bar{\psi} \cdot \nabla^\mu \nabla_\mu \phi + \nabla_\nu \nabla_\mu \bar{\phi} \bar{\epsilon} \gamma^\mu \gamma^\nu \psi + \nabla_\mu \bar{\phi} \nabla_\nu \bar{\epsilon} \gamma^\mu \gamma^\nu \psi \\
&\quad - \nabla^\mu \nabla_\mu \bar{\phi} \cdot \bar{\epsilon} \psi + \bar{\psi} \gamma^\mu \gamma^\nu \epsilon \nabla_\mu \nabla_\nu \phi + \bar{\psi} \gamma^\mu \gamma^\nu \nabla_\mu \epsilon \nabla_\nu \phi.
\end{align*}
\]

\[ \nabla_\mu \phi \equiv (\partial_\mu + iq V_\mu) \phi \quad \nabla_\mu \epsilon = \frac{i}{2f} \gamma_\mu \epsilon \]
SUSY Theories on squashed $S^3$

For simplicity, let's consider the free WZ model. We try

$$\delta \phi = \bar{\epsilon} \psi,$$

$$\delta \psi = i \gamma^\mu \epsilon \nabla_\mu \phi + \bar{\epsilon} F,$$

$$\delta F = i \epsilon \gamma^\mu \nabla_\mu \psi,$$

$$\mathcal{L} = \nabla_\mu \bar{\phi} \nabla^\mu \phi - i \bar{\psi} \gamma^\mu \nabla_\mu \psi + \bar{F} F.$$

$\partial_\mu$ of flat space theory was replaced by $\nabla_\mu$

$$\delta \mathcal{L} = \frac{iq}{2} V_{\mu \nu} \bar{\phi} \bar{\epsilon} \gamma^\mu \gamma^\nu \psi + \nabla_\mu \bar{\phi} \nabla_\nu \bar{\epsilon} \gamma^\mu \gamma^\nu \psi$$

$$+ \frac{iq}{2} V_{\mu \nu} \bar{\psi} \gamma^\mu \gamma^\nu \epsilon \phi + \bar{\psi} \gamma^\mu \gamma^\nu \nabla_\mu \epsilon \nabla_\nu \phi \neq 0.$$
The correct SUSY variation and Lagrangian

for chiral multiplet \((\phi, \psi, F)\) with U(1) R-charge \((-q, 1 - q, 2 - q)\) is,

\[
\delta \phi = \bar{\epsilon} \psi,
\]

\[
\delta \psi = i \gamma^\mu \epsilon \nabla_\mu \phi + \bar{\epsilon} F - \frac{q}{f} \epsilon \phi,
\]

\[
\delta F = i \epsilon \gamma^\mu \nabla_\mu \psi + \frac{2q-1}{2f} \epsilon \psi,
\]

\[
\mathcal{L} = \nabla_\mu \bar{\phi} \nabla^\mu \phi - i \bar{\psi} \gamma^\mu \nabla_\mu \psi + \bar{F} F
\]

\[
- \frac{(2q-1)}{2f} \bar{\psi} \psi + \left\{ \frac{qR}{4} - \frac{q(2q-1)}{2f^2} \right\} \bar{\phi} \phi
\]

Note:
SUSY \(\delta\) and Lagrangian \(\mathcal{L}\) for the theories on (squashed) \(S^3\) depends explicitly on \(q\) (R-charge assignment on matter fields).
Partition function \(Z\) also depends on \(q\).
This is made use of in “Z-minimization”.
Other Ingredients

Vector multiplet:

\[
\delta A_\mu = -\frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon),
\]

\[
\delta \sigma = \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon),
\]

\[
\delta \lambda = \frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma - \frac{1}{f}\sigma\epsilon,
\]

\[
\delta \bar{\lambda} = \frac{1}{2}\gamma^{\mu\nu}\bar{\epsilon} F_{\mu\nu} + D\bar{\epsilon} - i\gamma^\mu\bar{\epsilon} D_\mu\sigma + \frac{1}{f}\sigma\bar{\epsilon},
\]

\[
\delta D = -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon + \frac{i}{2}[\bar{\epsilon}\lambda + \bar{\lambda}\epsilon, \sigma]
\]

\[
-\frac{1}{4f}\bar{\epsilon}\lambda + \frac{1}{4f}\bar{\lambda}\epsilon
\]
Yang-Mills Lagrangian

\[ \mathcal{L}_{YM} = \text{Tr} \left( \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} D_\mu \sigma D^\mu \sigma + \frac{1}{2} \left( D + \frac{\sigma}{f} \right)^2 + \frac{i}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda + \frac{i}{2} \bar{\lambda} [\sigma, \lambda] - \frac{1}{4f} \bar{\lambda} \lambda \right) \]

Gauge-invariant Matter Kinetic Lagrangian

\[ \mathcal{L}_{\text{mat}} = D_\mu \bar{\phi} D^\mu \phi + \bar{\phi} \sigma^2 \phi + i \bar{\phi} D \phi + \bar{F} F + \frac{i(2q-1)}{f} \bar{\phi} \sigma \phi + \left( \frac{qR}{4} - \frac{q(2q-1)}{2f^2} \right) \bar{\phi} \phi \]

\[ -i \bar{\psi} \gamma^\mu D_\mu \psi + i \bar{\psi} \sigma \psi + i \bar{\psi} \lambda \phi - i \bar{\phi} \lambda \psi - \frac{(2q-1)}{2f} \bar{\psi} \psi. \]

See our paper for Chern-Simons, Fayet-Illiopoulos, Superpotential terms.
Calculation of Partition function

Strategy: Localization principle

1. Path integral for partition function localizes onto "saddle points", bosonic field configurations satisfying \( \delta(\text{fermion}) = 0 \)

2. \( L_{YM}, L_{\text{mat}} \) turn out to be SUSY exact.

   * \( \delta(\text{some fermion}) = L_{YM} = 0 \) at saddle points.

   \[
   F_{\mu\nu} = D_\mu \sigma = D + \frac{\sigma}{f} = 0.
   \]

   Saddle points are labelled by constant \( \sigma \).

   * \( L_{\text{mat}} \) is quadratic in matter fields, so \( \phi = F = 0 \) at saddle points.

\( L_{YM}, L_{\text{mat}} \) can be added to the Lagrangian (with arbitrary coefficients) without changing the value of partition function.
The following “saddle point approximation” gives an exact result for partition function.

\[
Z = \int d\sigma \left( \frac{\det(\Delta_F)}{\det(\Delta_B)} \right) \exp(-S_{\text{cl}})
\]

\(\Delta_B, \Delta_F\) are kinetic operators for bosons / fermions which are read from \(\mathcal{L}_{\text{YM}}, \mathcal{L}_{\text{mat}}\) in Gaussian approximation.

We found...
* for the squashed \(S^3\) with \(\text{SU}(2)_{\text{Left}}\) invariant metric, the determinant is the same as for round \(S^3\).
* for the squashed \(S^3\) with hyper-ellipsoidal metric, the determinant depends on
\[
b = \sqrt{\ell / \tilde{\ell}}
\]
1-loop determinant, SU(2)_Left invariant case

Notice that

\[ e^a \mu \partial_\mu = \left( \frac{1}{\ell} \mathcal{R}^1, \frac{1}{\ell} \mathcal{R}^2, \frac{1}{\ell} \mathcal{R}^3 \right) \]

\[ \mathcal{R}^a : \text{ Vector fields generating SU(2)_Right action} \]

**Matter determinant.**

For simplicity, we consider an electron chiral multiplet of R-charge \( q \) which is charged (+1) under an abelian vectormultiplet.

Kinetic operator for boson \( \phi \) and fermion \( \psi \) read

\[
\Delta_\phi = \frac{4}{\ell^2} (J^1 J^1 + J^2 J^2) + \frac{4}{\ell^2} \left( J^3 + \frac{q}{2} \left( 1 - \frac{\ell}{f} \right) \right)^2 + \sigma^2 + \frac{2i(q - 1)\sigma}{f} - \frac{q^2}{f^2} + \frac{2q}{f\ell}.
\]

\[
\Delta_\psi = \frac{4}{\ell} (S^1 J^1 + S^2 J^2) + \frac{4}{\ell} S^3 J^3 + \frac{1}{\ell} + \frac{1 - q}{f} + 2(q - 1) \left( \frac{1}{\ell} - \frac{1}{f} \right) S^3
\]

\[ J^a = \frac{1}{2i} \mathcal{R}^a, \quad S^a = \frac{1}{2} \gamma^a, \quad f = \frac{\ell^2}{\ell} \]
Matter determinant: final form (* after cancellation of many eigenvalues!)

\[
\frac{\text{det} \Delta_{\psi}}{\text{det} \Delta_{\phi}} = \prod_{n>0} \left( \frac{n + 1 - q + i \tilde{\ell} \sigma}{n - 1 + q - i \tilde{\ell} \sigma} \right)^n = s_{b=1}(i - iq - \tilde{\ell} \sigma)
\]

Essentially the same as for the round $S^3$.

$n$ has the meaning $n = 2j + 1$, (orbital angular momentum $(j, j)$)

Degeneration of zeroes and poles is due to unbroken SU(2)$_{\text{Left}}$.

So, to find the generalization to $b \neq 1$, we need to look for less symmetric squashings.
**Vectormultiplet determinant**

We decompose the vector multiplet fields into Cartan-Weyl basis, eg

\[
\lambda = \sum_i \lambda_i H_i + \sum_{\alpha \in \Delta_+} (\lambda_\alpha E_\alpha + \lambda_{-\alpha} E_{-\alpha})
\]

At the saddle point labelled by \( \sigma \), \((A_\alpha, \lambda_\alpha, \varphi_\alpha)\) acquire \((\text{mass})^2 \sim (\sigma \cdot \alpha)^2\)

\(\varphi = \) (quantum fluctuation of the scalar around saddle point \( \sigma \))

\[
\left( \frac{\det \Delta_{\lambda} }{ \det \Delta_{A, \varphi} } \right) = \prod_{\alpha \in \Delta_+} \left( \frac{\det \Delta_{\lambda_\alpha} }{ \det \Delta_{A_\alpha, \varphi_\alpha} } \right)^2
\]

\(\det \Delta_{\lambda_\alpha} : \text{same as matter fermions.}\)

\(\det \Delta_{A_\alpha, \varphi_\alpha} : \text{complicated, since } A_\alpha \text{ and } \varphi_\alpha \text{ mix.}\)
Calculation of $\det \Delta A_\alpha, \varphi_\alpha$:

First, consider the 4 modes (with mode-variables $x_+, x_-, x_3, x$)

\[
A_\alpha = x_+ Y_{j,n,m-1} \mu^+ + x_3 Y_{j,n,m} \mu^3 + x_- Y_{j,n,m+1} \mu^- ,
\]
\[
\varphi_\alpha = x Y_{j,n,m}
\]

$Y_{j,n,m}$, $Y_{j_L,n,m}$, $Y_{j_R,n,m}$ : spherical harmonics, $\mu^a$ : LI 1-forms

Then $\Delta A_\alpha, \varphi_\alpha$ mixes these four modes among themselves, but not with anything else.
Calculation of $\det \Delta A_\alpha, \varphi_\alpha$:

The **4 modes** split into

* 2 longitudinal modes: $A_\alpha \sim d\varphi_\alpha$
* 2 transverse modes: $\varphi_\alpha = d \ast A_\alpha = 0$

The 2 longitudinal modes have eigenvalues

$\Delta A_\alpha, \varphi_\alpha = 0$

**(gauge mode)**

$\Delta A_\alpha, \varphi_\alpha = \frac{4j(j + 1) - m^2}{\ell^2} + \frac{4m^2}{\tilde{\ell}^2} + (\sigma \cdot \alpha)^2$

**(cancel with the eigenvalues in FP determinant)**
Vectormultiplet determinant: final result

\[ \int_G d\sigma \left( \frac{\det \Delta_\lambda}{\det \Delta_{A,\varphi}} \right) = \int_G d\sigma \prod_{\alpha \in \Delta_+} \left( \frac{\sinh(\pi \tilde{\ell} \sigma \cdot \alpha)}{\pi \tilde{\ell} \sigma \cdot \alpha} \right)^2 \]

\[ = \int_T d^r \sigma \prod_{\alpha \in \Delta_+} \sinh^2(\pi \tilde{\ell} \sigma \cdot \alpha) \]

Again, essentially the same as for round $S^3$. 
1-loop determinant, Hyper-ellipsoid case

\[
\begin{align*}
\int\propto^2 &= f^2 \propto^2 + \ell^2 \cos^2 \theta \propto^2 + \tilde{\ell}^2 \sin^2 \theta \propto^2 \\
f &\equiv \sqrt{\ell^2 \sin^2 \theta + \tilde{\ell}^2 \cos^2 \theta} \\
\epsilon &= \frac{1}{\sqrt{2}} \begin{pmatrix}
-e^{i/2}(\chi-\varphi+\theta) \\
e^{-i/2}(\chi-\varphi-\theta)
\end{pmatrix}, \quad \bar{\epsilon} = \frac{1}{\sqrt{2}} \begin{pmatrix}
e^{i/2}(-\chi+\varphi+\theta) \\
\frac{i}{2}(\chi-\varphi-\theta)
\end{pmatrix}
\end{align*}
\]

There is only U(1)xU(1) symmetry.

It is too difficult to find out all the eigenmodes. \(\Rightarrow\) We need a different route.

We recall

Due to SUSY, most of the eigenvalues cancel out between bosons and fermions.

Non-trivial contributions to determinant arise from "unpaired modes".
Matter determinant

for a electron chiral multiplet charged (+1)
under an abelian vectormultiplet.

We take as the regulator Lagrangian

\[ \mathcal{L}_{\text{reg}} = \delta_c \delta_{\bar{c}} (\overline{\psi} \psi - 2i \phi \sigma \phi) \]

= (slightly different from \( \mathcal{L}_{\text{mat}} \))

and study the spectrum of kinetic operators \( \Delta_\phi, \Delta_\psi \).
Multiplet structure of eigenmodes: we found

1. \[ \Delta_p \Psi = M \Psi \quad \Phi \equiv \bar{\epsilon} \Psi, \]
   \[ \rightarrow \Delta_\phi \Phi = M(M - 2i\sigma)\Phi \]

2. \[ \Delta_\phi \Phi = M(M - 2i\sigma)\Phi \]

\[
\begin{cases}
\Psi_1 \equiv \epsilon \Phi,
\Psi_2 \equiv i\gamma^\mu \epsilon D_\mu \Phi + i\epsilon \sigma \Phi - \frac{q}{f} \epsilon \Phi
\end{cases}
\]

\[ \rightarrow \begin{pmatrix}
\Delta_p \Psi_1 \\
\Delta_p \Psi_2
\end{pmatrix} = \begin{pmatrix}
2i\sigma & -1 \\
-M(M - 2i\sigma) & 0
\end{pmatrix} \begin{pmatrix}
\Psi_1 \\
\Psi_2
\end{pmatrix} \]

So, one scalar mode \( \Phi : \Delta_\phi = M(M - 2i\sigma) \)

and two spinor modes \( \Psi_1, \Psi_2 : \Delta_p = M, 2i\sigma - M \) form a multiplet.
Nontrivial contributions to determinant arise from

1. unpaired spinor eigenmode

\[ \Delta_\psi \Psi = M \Psi \quad \text{but} \quad \bar{c} \Psi = 0. \]

\[ \ldots \text{contribute } M \text{ to the enumerator of determinant.} \]

2. missing spinor eigenmode

\[ \Psi_1 \equiv \epsilon \Phi, \]
\[ \Psi_2 \equiv i \gamma^\mu \epsilon D_\mu \Phi + i \epsilon \sigma \Phi - \frac{q}{f} \epsilon \Phi \]
\[
\left\{ \begin{array}{l}
\Delta_\psi \Psi_1 = (2i \sigma - M) \Psi_1 \\
\Delta_\phi \Phi = M(M - i \sigma) \Phi.
\end{array}\right. \]

One can show that if \( \Psi_2 = M \Psi_1 \),

One can find these cases by solving simple 1\textsuperscript{st} order differential equations.
Matter determinant:

\[
\left( \frac{\det \Delta_\psi}{\det \Delta_\phi} \right) = \frac{\prod \text{(unpaired spinor eigenvalues)}}{\prod \text{(missing spinor eigenvalues)}}
\]

\[
= \prod_{m,n \geq 0} \frac{\frac{m}{\ell} + \frac{n}{\ell} + i\sigma - \frac{q-2}{2} \left( \frac{1}{\ell} + \frac{1}{\tilde{\ell}} \right)}{\frac{m}{\ell} + \frac{n}{\ell} - i\sigma + \frac{q}{2} \left( \frac{1}{\ell} + \frac{1}{\tilde{\ell}} \right)}
\]

\[
= s_b \left( \frac{iQ}{2} (1 - q) - i\hat{\sigma} \right)
\]

where

\[
Q = b + \frac{1}{b}, \quad b = \sqrt{\ell/\tilde{\ell}}, \quad \hat{\sigma} = \sqrt{\ell\tilde{\ell}\sigma}.
\]
Concluding Remarks

Generalizations to higher-dimensional spheres and their deformations will be interesting.

4d “squashed sphere” . . . Generalization of (original) AGT relation to \( b \neq 1 \)

Five-sphere . . . M5-brane(?)

In higher dimensions, correct generalization of Killing spinor equation is determined from **off-shell supergravity**

(cf. Festuccia-Seiberg 1105.0689)